

10 Fourier Series

10.1 Introduction

When the French mathematician Joseph Fourier (1768-1830) was trying to study the flow of heat in a metal plate, he had the idea of expressing the heat source as an infinite series of sine and cosine functions. Although the original motivation was to solve the heat equation, it later became obvious that the same techniques could be applied to a wide array of mathematical and physical problems.

In this course, we will learn how to find Fourier series to represent *periodic* functions as an infinite series of sine and cosine terms.

A function $f(x)$ is said to be periodic with period T , if

$$f(x + T) = f(x), \text{ for all } x.$$

The period of the function $f(t)$ is the interval between two successive repetitions.

10.2 Definition of a Fourier Series

Let f be a bounded function defined on the $[-\pi, \pi]$ with at most a finite number of maxima and minima and at most a finite number of discontinuities in the interval. Then the Fourier series of f is the series

$$\begin{aligned} f(x) = \frac{1}{2}a_0 &+ a_1 \cos x + a_2 \cos 2x + a_3 \cos 3x + \dots \\ &+ b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x + \dots \end{aligned}$$

where the coefficients a_n and b_n are given by the formulae

$$\begin{aligned} a_0 &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx \\ a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) dx \\ b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) dx \end{aligned}$$

10.3 Why the coefficients are found as they are

We want to derive formulas for a_0 , a_n and b_n . To do this we take advantage of some properties of sinusoidal signals. We use the following orthogonality conditions:

Orthogonality conditions

- (i) The average value of $\cos(nx)$ and $\sin(nx)$ over a period is zero.

$$\int_{-\pi}^{\pi} \cos(nx) dx = 0$$

$$\int_{-\pi}^{\pi} \sin(nx) dx = 0$$

- (ii) The average value of $\sin(mx)\cos(nx)$ over a period is zero.

$$\int_{-\pi}^{\pi} \sin(mx)\cos(nx) dx = 0$$

- (iii) The average value of $\sin(mx)\sin(nx)$ over a period,

$$\int_{-\pi}^{\pi} \sin(mx)\sin(nx) dx = \begin{cases} \pi & \text{if } m = n \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

- (iv) The average value of $\cos(mx)\cos(nx)$ over a period,

$$\int_{-\pi}^{\pi} \cos(mx)\cos(nx) dx = \begin{cases} 2\pi & \text{if } m = n = 0 \\ \pi & \text{if } m = n \neq 0 \\ 0 & \text{if } m \neq n \end{cases}$$

REMARK. The following trigonometric identities are useful to prove the above orthogonality conditions:

$$\cos A \cos B = \frac{1}{2}[\cos(A - B) + \cos(A + B)]$$

$$\sin A \sin B = \frac{1}{2}[\cos(A - B) - \cos(A + B)]$$

$$\sin A \cos B = \frac{1}{2}[\sin(A - B) + \sin(A + B)]$$

Finding a_n

We assume that we can represent the function by

$$\begin{aligned} f(x) = \frac{1}{2}a_0 &+ a_1 \cos x + a_2 \cos 2x + a_3 \cos 3x + \dots \\ &+ b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x + \dots \end{aligned} \quad (1)$$

Multiply (1) by $\cos(nx)$, $n \geq 1$ and integrate from $-\pi$ to π and assume it is permissible to integrate the series term by term.

$$\begin{aligned} \int_{-\pi}^{\pi} f(x) \cos(nx) dx &= \frac{1}{2}a_0 \int_{-\pi}^{\pi} \cos(nx) dx + a_1 \int_{-\pi}^{\pi} \cos x \cos(nx) dx + a_2 \int_{-\pi}^{\pi} \cos 2x \cos(nx) dx + \dots \\ &+ b_1 \int_{-\pi}^{\pi} \sin x \cos(nx) dx + b_2 \int_{-\pi}^{\pi} \sin 2x \cos(nx) dx + \dots \\ &= a_n \pi, \text{ because of the above orthogonality conditions} \\ a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) dx \end{aligned}$$

Similarly if we multiply (1) by $\sin(nx)$, $n \geq 1$ and integrate from $-\pi$ to π , we can find the formula for the coefficient b_n . To find a_0 , simply integrate (1) from $-\pi$ to π .

10.4 Fourier Series

DEFINITION. Let $f(x)$ be a 2π -periodic function which is integrable on $[-\pi, \pi]$. Set

$$\begin{aligned}a_0 &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx \\a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) dx \\b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) dx\end{aligned}$$

then the trigonometric series

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx))$$

is called the Fourier series associated to the function $f(x)$.

REMARK. Notice that we are not saying $f(x)$ is equal to its Fourier Series. Later we will discuss conditions under which that is actually true.

EXAMPLE. Find the Fourier coefficients and the Fourier series of the square-wave function f defined by

$$f(x) = \begin{cases} 0 & \text{if } -\pi \leq x < 0 \\ 1 & \text{if } 0 \leq x < \pi \end{cases} \quad \text{and} \quad f(x + 2\pi) = f(x)$$

SOLUTION: So f is periodic with period 2π and its graph is:

Using the formulas for the Fourier coefficients we have

$$\begin{aligned}a_0 &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx \\&= \frac{1}{\pi} \left(\int_{-\pi}^0 0 dx + \frac{1}{\pi} \int_0^{\pi} 1 dx \right) \\&= \frac{1}{\pi} (\pi) \\&= 1 \\a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) dx \\&= \frac{1}{\pi} \left(\int_{-\pi}^0 0 dx + \frac{1}{\pi} \int_0^{\pi} \cos(nx) dx \right) \\&= \frac{1}{\pi} \left(0 + \left[\frac{\sin(nx)}{n} \right]_0^{\pi} \right) \\&= \frac{1}{n\pi} (\sin n\pi - \sin 0) \\&= 0 \\b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) dx \\&= \frac{1}{\pi} \left(\int_{-\pi}^0 0 dx + \frac{1}{\pi} \int_0^{\pi} \sin(nx) dx \right) \\&= \frac{1}{\pi} \left(- \left[\frac{\cos(nx)}{n} \right]_0^{\pi} \right) \\&= -\frac{1}{n\pi} (\cos n\pi - \cos 0) \\&= -\frac{1}{n\pi} ((-1)^n - 1), \quad \text{since } \cos n\pi = (-1)^n. \\&= \begin{cases} 0, & \text{if } n \text{ is even} \\ \frac{2}{n\pi}, & \text{if } n \text{ is odd} \end{cases}\end{aligned}$$

The Fourier Series of f is therefore

$$\begin{aligned}
 f(x) &= \frac{1}{2}a_0 + a_1 \cos x + a_2 \cos 2x + a_3 \cos 3x + \dots \\
 &\quad + b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x + \dots \\
 &= \frac{1}{2} + 0 + 0 + 0 + \dots \\
 &\quad + \frac{2}{\pi} \sin x + 0 \sin 2x + \frac{2}{3\pi} \sin 3x + 0 \sin 4x + \frac{2}{5\pi} \sin 5x + \dots \\
 &= \frac{1}{2} + \frac{2}{\pi} \sin x + \frac{2}{3\pi} \sin 3x + \frac{2}{5\pi} \sin 5x + \dots
 \end{aligned}$$

REMARK. In the above example we have found the Fourier Series of the square-wave function, but we don't know yet whether this function is equal to its Fourier series. If we plot

$$\begin{aligned}
 &\frac{1}{2} + \frac{2}{\pi} \sin x \\
 &\frac{1}{2} + \frac{2}{\pi} \sin x + \frac{2}{3\pi} \sin 3x \\
 &\frac{1}{2} + \frac{2}{\pi} \sin x + \frac{2}{3\pi} \sin 3x + \frac{2}{5\pi} \sin 5x + \\
 &\frac{1}{2} + \frac{2}{\pi} \sin x + \frac{2}{3\pi} \sin 3x + \frac{2}{5\pi} \sin 5x + \frac{2}{7\pi} \sin 7x + \\
 &\frac{1}{2} + \frac{2}{\pi} \sin x + \frac{2}{3\pi} \sin 3x + \frac{2}{5\pi} \sin 5x + \frac{2}{7\pi} \sin 7x + \frac{2}{9\pi} \sin 9x + \frac{2}{11\pi} \sin 11x + \frac{2}{13\pi} \sin 13x,
 \end{aligned}$$

we see that as we take more terms, we get a better approximation to the square-wave function. The following theorem tells us that for almost all points (except at the discontinuities), the Fourier series equals the function.

THEOREM (FOURIER CONVERGENCE THEOREM) *If f is a periodic function with period 2π and f and f' are piecewise continuous on $[-\pi, \pi]$, then the Fourier series is convergent. The sum of the Fourier series is equal to $f(x)$ at all numbers x where f is continuous. At the numbers x where f is discontinuous, the sum of the Fourier series is the average value. i.e.*

$$\frac{1}{2}[f(x^+) + f(x^-)].$$

REMARK. If we apply this result to the example above, the Fourier Series is equal to the function at all points except $-\pi, 0, \pi$. At the discontinuity 0, observe that

$$f(0^+) = 1 \quad \text{and} \quad f(0^-) = 0$$

The average value is $1/2$.

Therefore the series equals to $1/2$ at the discontinuities.

10.5 Odd and even functions

DEFINITION. A function is said to be even if

$$f(-x) = f(x) \quad \text{for all real numbers } x.$$

A function is said to be odd if

$$f(-x) = -f(x) \quad \text{for all real numbers } x.$$

EXAMPLE. $\cos x$, x^2 , $|x|$ are examples of even functions. $\sin x$, x , x^3 are examples of odd functions. The product of two even functions is even, the product of two odd functions is also even. The product of an even and odd function is odd.

REMARK. If f is an **odd** function then

$$\int_{-\pi}^{\pi} f(x)dx = 0,$$

while if f is an **even** function then

$$\int_{-\pi}^{\pi} f(x)dx = 2 \int_0^{\pi} f(x)dx.$$

(i) If $f(x)$ is odd, then

- $f(x)\cos(nx)$ is odd hence $a_n = 0$ and
- $f(x)\sin(nx)$ is even hence $b_n = \frac{2}{\pi} \int_0^{\pi} f(x)\sin(nx)dx$

(ii) If $f(x)$ is even, then

- $f(x)\cos(nx)$ is even hence $a_n = \frac{2}{\pi} \int_0^{\pi} f(x)\cos(nx)dx$ and
- $f(x)\sin(nx)$ is odd hence $b_n = 0$

EXAMPLE. 1. Let f be a periodic function of period 2π such that

$$f(x) = x \quad \text{for } -\pi \leq x < \pi$$

Find the Fourier series associated to f .

SOLUTION: So f is periodic with period 2π and its graph is:

We first check if f is even or odd.

$$f(-x) = -x = -f(x), \quad \text{so } f(x) \text{ is odd.}$$

Therefore,

$$\begin{aligned} a_n &= 0 \\ b_n &= \frac{2}{\pi} \int_0^\pi f(x) \sin(nx) dx \end{aligned}$$

Using the formulas for the Fourier coefficients we have

$$\begin{aligned} b_n &= \frac{2}{\pi} \int_0^\pi x \sin(nx) dx \\ &= \frac{2}{\pi} \left(\left[-x \frac{\cos nx}{n} \right]_0^\pi - \int_0^\pi \left(-\frac{\cos nx}{n} \right) dx \right) \\ &= \frac{2}{\pi} \left(\frac{1}{n} [-\pi \cos n\pi] + \left[\frac{\sin nx}{n^2} \right]_0^\pi \right) \\ &= -\frac{2}{n} \cos n\pi \\ &= \begin{cases} -2/n & \text{if } n \text{ is even} \\ 2/n & \text{if } n \text{ is odd} \end{cases} \end{aligned}$$

The Fourier Series of f is therefore

$$\begin{aligned} f(x) &= b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x + \dots \\ &= 2 \sin x - \frac{2}{2} \sin 2x + \frac{2}{3} \sin 3x - \frac{2}{4} \sin 4x + \frac{2}{5} \sin 5x + \dots \end{aligned}$$

2. Let f be a periodic function of period 2π such that

$$f(x) = \pi^2 - x^2 \quad \text{for } -\pi \leq x < \pi$$

SOLUTION: So f is periodic with period 2π and its graph is:

We first check if f is even or odd.

$$f(-x) = \pi^2 - (-x)^2 = \pi^2 - x^2 = f(x), \quad \text{so } f(x) \text{ is even.}$$

Since f is even,

$$b_n = 0$$
$$a_n = \frac{2}{\pi} \int_0^\pi f(x) \cos(nx) dx$$

Using the formulas for the Fourier coefficients we have

$$\begin{aligned}
 a_n &= \frac{2}{\pi} \int_0^\pi f(x) \cos(nx) dx \\
 &= \frac{2}{\pi} \int_0^\pi (\pi^2 - x^2) \cos(nx) dx \\
 &= \frac{2}{\pi} \left(\left[(\pi^2 - x^2) \frac{\sin nx}{n} \right]_0^\pi - \int_0^\pi -2x \frac{\sin nx}{n} dx \right) \\
 &= \frac{2}{\pi} \left(\left[(\pi^2 - \pi^2) \frac{\sin n\pi}{n} - (\pi^2 - 0) \frac{\sin 0}{n} \right] + \int_0^\pi 2x \frac{\sin nx}{n} dx \right) \\
 &= \frac{2}{\pi} \frac{2}{n} \int_0^\pi x \sin(nx) dx \\
 &= \frac{2}{\pi} \frac{2}{n} \left(\left[-x \frac{\cos nx}{n} \right]_0^\pi - \int_0^\pi \left(-\frac{\cos nx}{n} \right) dx \right) \\
 &= \frac{2}{\pi} \frac{2}{n} \frac{1}{n} \left([-\pi \cos n\pi] + \left[\frac{\sin nx}{n^2} \right]_0^\pi \right) \\
 &= -\frac{4}{n^2} \cos n\pi \\
 &= \begin{cases} -4/n^2 & \text{if } n \text{ is even} \\ 4/n^2 & \text{if } n \text{ is odd} \end{cases}
 \end{aligned}$$

It remains to calculate a_0 .

$$\begin{aligned}
 a_0 &= \frac{2}{\pi} \int_0^\pi (\pi^2 - x^2) dx \\
 &= \frac{2}{\pi} \left[\pi^2 x - \frac{x^3}{3} \right]_0^\pi \\
 &= \frac{4\pi^2}{3}
 \end{aligned}$$

The Fourier Series of f is therefore

$$\begin{aligned}
 f(x) &= \frac{1}{2} a_0 + a_1 \cos x + a_2 \cos 2x + a_3 \cos 3x + \dots \\
 &\quad b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x + \dots \\
 &= \frac{2\pi^2}{3} + 4 \left(\cos x - \frac{1}{4} \cos 2x + \frac{1}{9} \cos 3x - \frac{1}{16} \cos 4x + \frac{1}{25} \cos 5x + \dots \right)
 \end{aligned}$$